Practice Exam 1 — Functional Analysis (WIFA–08)

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.

Problem 1

Consider the following normed linear space:

$$\ell^1 = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, \sum_{i=1}^{\infty} |x_i| < \infty \right\},$$

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|.$$

(a) Prove that ℓ^1 provided with the norm $\|\cdot\|_1$ is a Banach space.

(b) Prove that $\|x\|_s = \sup_{n\in\mathbb{N}} \left| \sum_{i=1}^{n} x_i \right|$ is also a norm on ℓ^1.

(c) Prove that $\|x\|_s \leq \|x\|_1$ for all $x \in \ell^1$.

(d) Are the norms $\|\cdot\|_1$ and $\|\cdot\|_s$ equivalent?

Problem 2

Consider the following linear operator:

$$S_r : \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

Prove the following statements:

(a) $\|S_r\| = 1$;

(b) S_r is *not* compact;

(c) S_r has no eigenvalues;

(d) $|\lambda| < 1$ implies that $\operatorname{ran}(S_r - \lambda)$ is not dense in ℓ^2;

(e) $\sigma(S_r) = \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq 1\}$.

Problem 3

(a) Formulate the closed graph theorem.

(b) Let X be a Banach space and let $V, W \subset X$ be closed linear subspaces such that $X = V + W$ is a direct sum. This means that each $x \in X$ can be uniquely written as $x = v + w$ with $v \in V$ and $w \in W$.

Prove the following statements:

(i) $P : X \to X$ defined by $Px = v$ is a linear map;

(ii) P is a projection;

(iii) P is bounded.

Problem 4

(a) Formulate the Hahn-Banach theorem for normed linear spaces.

(b) Let X be a normed linear space and let $V \subset X$ be a linear subspace. Prove that the following statements are equivalent:

(i) V is dense in X;

(ii) $f \in X'$ with $f(v) = 0$ for all $v \in V$ implies that $f = 0$.

End of test

Solution of Problem 1

(a) Let x^n be a Cauchy sequence in ℓ^1. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Rightarrow \quad |x_i^n - x_i^m| \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m| = \|x^n - x^m\|_1 \leq \varepsilon.$$

In particular, (x_i^n) is a Cauchy sequence in $\mathbb{K}$ for all $i \in \mathbb{N}$. Since $\mathbb{K}$ is complete $x_i^n \to x_i$ for some $x_i \in \mathbb{K}$. Define

$$x = (x_1, x_2, x_3, \dots)$$

and let $p \in \mathbb{N}$ be arbitrary. Then

$$\begin{aligned} m, n \geq N \quad &\Rightarrow \quad \sum_{i=1}^{p} |x_i^n - x_i^m| \leq \varepsilon \\ &\Rightarrow \quad \sum_{i=1}^{p} |x_i^n - x_i| \leq \varepsilon \quad (\text{let } m \to \infty) \\ &\Rightarrow \quad \sum_{i=1}^{\infty} |x_i^n - x_i| \leq \varepsilon \quad (\text{let } p \to \infty) \\ &\Rightarrow \quad \|x^n - x\|_1 \leq \varepsilon \end{aligned}$$

which shows that $x^n \to x$ in ℓ^1. In addition, since $x^N \in \ell^1$ and $x^N - x \in \ell^2$ it follows that $x = x^N - (x^N - x) \in \ell^1$.

(b) Clearly $\|x\|_s \geq 0$ and

$$\begin{aligned} \|x\|_s = 0 \quad &\Rightarrow \quad \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x_i \right| = 0 \\ &\Rightarrow \quad \left| \sum_{i=1}^{n} x_i \right| = 0 \quad \forall n \in \mathbb{N} \\ &\Rightarrow \quad |x_1| = 0, \quad |x_1 + x_2| = 0, \quad |x_1 + x_2 + x_3| = 0, \dots \\ &\Rightarrow \quad x = 0. \end{aligned}$$

If $\lambda \in \mathbb{K}$ then

$$\|\lambda x\|_s = \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} \lambda x_i \right| = \sup_{n \in \mathbb{N}} |\lambda| \left| \sum_{i=1}^{n} x_i \right| = |\lambda| \, \|x\|_s.$$

Finally, the triangle inequality is proven as follows:

$$\|x + y\|_s = \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} (x_i + y_i) \right| \leq \sup_{n \in \mathbb{N}} \left\{ \left| \sum_{i=1}^{n} x_i \right| + \left| \sum_{i=1}^{n} y_i \right| \right\} \leq \|x\|_s + \|y\|_s.$$

(c) This follows from:

$$\|x\|_s = \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x_i \right| \leq \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |x_i| = \|x\|_1.$$

(d) The norms $\|\cdot\|_1$ and $\|\cdot\|_s$ are not equivalent. Indeed, take the sequence

$$x^n = \Big(\underbrace{\frac{1}{n}, -\frac{1}{n}, \ldots, \frac{1}{n}, -\frac{1}{n}}_{n \text{ times}}, 0, 0, 0, \ldots\Big)$$

then $\|x^n\|_s = 1/n \to 0$ whereas $\|x^n\|_1 = 2$ for all $n \in \mathbb{N}$.

Solution of Problem 2

(a) For every $x \in \ell^2$ we have $\|S_r x\|_2 = \|x\|_2$ which implies that

$$\|S_r\| = \sup_{x \neq 0} \frac{\|S_r x\|_2}{\|x\|_2} = 1.$$

(b) Let $x^n \in \ell^2$ the n-th unit vector (i.e., zeros everywhere, except for a 1 at the n-th position). Then (x^n) is bounded in ℓ^2 as $\|x^n\| = 1$ for all $n \in \mathbb{N}$ but

$$\|S_r x^n - S_r x^m\| = \sqrt{2}, \quad n \geq m.$$

This implies that $(S_r x^n)$ does not have a convergent subsequence. Hence, S_r is not compact.

(c) Assume that $S_r x = \lambda x$. Since S_r is isometric (see part a), it follows that S_r is injective and thus $\lambda \neq 0$. Note that

$$(0, x_1, x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots),$$

which implies that $x = 0$. Hence, S_r has no eigenvalues.

(d) If $|\lambda| < 1$ then $y = (1, \bar{\lambda}, \bar{\lambda}^2, \ldots) \in \ell^2$. Note that

$$y \perp (S_r - \lambda)x = (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \ldots) \in \operatorname{ran}(S_r - \lambda)$$

for all $x \in \ell^2$. This implies that $\operatorname{ran}(S_r - \lambda)$ is not dense in ℓ^2 which shows that $\lambda \in \sigma(S_r)$.

(e) From part (b) it follows that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(S_r)$. Since the spectrum is closed, it also follows that $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \sigma(S_r)$. On the other hand, if $|\lambda| > 1 = \|S_r\|$, then it follows that $\lambda \in \rho(S_r)$. We conclude that $\sigma(S_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Solution of Problem 3

(a) Assume that X and Y are Banach spaces, $U \subset X$ is a closed linear subspace, and $T : U \to Y$ is a linear operator. If the graph $G(T)$ of T is closed then $T \in B(U, Y)$.

(b) (i) Let $x, y \in X$, then there exist unique $v_1, v_2 \in V$ and $w_1, w_2 \in W$ such that $x = v_1 + w_1$ and $y = v_2 + w_2$. Then

$$\lambda x + \mu y = (\lambda v_1 + \mu v_2) + (\lambda w_1 + \mu w_2)$$

where $\lambda v_1 + \mu v_2 \in V$ and $\lambda w_1 + \mu w_2 \in W$. Hence,

$$P(\lambda x + \mu y) = \lambda v_1 + \mu v_2 = \lambda P x + \mu P y,$$

which shows that P is indeed a linear map.

(ii) Write $x = v + w$ with $v \in V$ and $w \in W$, then $Px = v = v + 0$ and the latter decomposition is unique. Hence, $P^2 x = Pv = v = Px$ which shows that $P^2 = P$ so that P is indeed a projection.

(iii) Let $(x, y) \in \overline{G(P)}$ then there exists a sequence $(x_n, Px_n) \to (x, y)$. In particular, it follows that $x_n \to x$ in X and $Px_n \to y$ in X. Since $Px_n \in V$ for all $n \in \mathbb{N}$ and V is closed, it follows that $y \in V$ as well. Since $x_n - Px_n \in W$ for all $n \in \mathbb{N}$, $x_n - Px_n \to x - y$, and W is closed, it follows that $x - y \in W$. Hence, $Px - y = P(x - y) = 0$ so that $y = Px$. This shows that $(x, y) \in G(P)$ and we conclude that P is closed. Applying the closed graph theorem with $U = X = Y$ shows that P is bounded.

Solution of Problem 4

(a) Let X be a normed linear space and let $V \subset X$ be a linear subspace. If $f \in V'$ then there exists $F \in X'$ such that $F(v) = f(v)$ for all $v \in V$ and $\|F\| = \|f\|$.

(b) Assume that V is dense in X. Let $x \in X$, then there exists $v_n \in V$ such that $v_n \to x$. If $f \in X'$ satisfies $f(v) = 0$ for all $v \in V$, then

$$|f(x)| = |f(x - v_n) + f(v_n)| = |f(x - v_n)| \leq \|f\| \, \|x - v_n\| \to 0$$

which shows that $f(x) = 0$. Since $x \in X$ was arbitrary it follows that $f = 0$.

For the converse, assume that V is not dense in X. Pick $x_0 \in X \setminus \overline{V}$ and define the linear functional

$$g : \operatorname{span}\{V, x_0\} \to \mathbb{K}, \quad g(v + \lambda x_0) = \lambda.$$

By the Hahn-Banach theorem there exists $G \in X'$ which extends g to all of X. Clearly, $G(v) = 0$ for all $v \in V$ while $G \neq 0$ as $G(x_0) = 1$. This leads to a contradiction. Hence, V must be dense in X.